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ON THE FUNDAMENTAL FREQUENCY OF BEAMS CARRYING A POINT MASS: RAYLEIGH APPROXIMATIONS VERSUS EXACT SOLUTIONS

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1. INTRODUCTION

Point mass–beam systems are often used as first approximation models for a variety of structural and machine elements. Hence, one frequently faces the task of determining the natural frequencies of such systems. Although no unsurmountable difficulty arises neither in the determination nor in the solution of the exact frequency equations of these models, there are many situations where more than an approximate knowledge of just the fundamental frequency is hardly necessary. In such cases, approximate formulae obtained by using the Rayleigh method are known to constitute good alternatives. It is, however, also known that the performance of such formulae closely depend on the choice of the substitutes used for the mode shapes of the system in question. Rayleigh himself, when considering the problem of determining the fundamental frequency of a uniform cantilever beam carrying a tip mass, as an illustrative example of his method [1, V.1, p. 289], has used the static deflection curve of the beam acted upon by a concentrated tip load, as a good estimate of the fundamental mode shape (actually the exact one for the limiting case of a massless beam), and derived now is the well-known formula

$$\omega \cong \sqrt{\frac{\mathrm{EI}}{\mu\ell^4}}\sqrt{\frac{3}{[(33/140)+\gamma]}}, \tag{1}$$

where EI is the flexural rigidity, ℓ the length, μ the mass per unit length of the beam and γ is the ratio of the tip mass to the beam mass. Along the same lines, i.e., applying the Rayleigh method in conjunction with the static deflection curve due to the effect of a concentrated load applied at the location of the point mass, Timoshenko [2] and others have provided the practising engineers with a series of useful formulae corresponding to various beam-point mass configurations. These formulae behave fairly accurately in the cases where the inertial effect of the point mass dominates that of the beam, i.e., when the mass ratio γ is large and the location of the point mass is at or near the maximal amplitude point. But they lose their

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accuracy with inc r easing inertial contribution of the beam, because so does the used mode shape estimate. On the other hand, in the limiting case of a bare beam, the static deflection curve due to the effect of the beam's own weight is known to constitute a good estimate for the fundamental mode shape. One may therefore infer that static deflection curves corresponding to the combined effect of the beam and point mass weights would constitute better choices for beam–point mass systems as they would provide a certain flexibility in adapting to the fundamental mode shapes in a broader range of the mass ratio γ and of the point mass location. This kind of shape estimate was actually applied by Humar [3] to centrally loaded, simply supported beams and by Low [4] to simply supported, fixed–fixed and cantilever beams carrying an arbitrarily located point mass, and has favourably been compared in reference [4] to three other kinds of shape estimates through a comparison with an existing set of exact solutions. The Rayleigh formulae obtained in reference [4] were, however, rather unwieldy and the problem of determining their validity limits remained.

The purpose of the present note is to reconsider this problem for beams with various classical end conditions and to show that the resulting formulae can be put in reasonably simple forms in the special cases where the beam is symmetrically supported. The corresponding exact frequency equation is also given for each case and the results are compared in a broad range of the relevant parameters so that a clear idea on the validity limits of the formulae follows.

2. THEORETICAL BACKGROUND

Consider an Euler–Bernouilli beam with unspecified end conditions carrying a point mass M (Figure 1). Let the lateral motions of the beam points at the left and right of M be represented by $y_1(u, t)$; $0 \leqslant u \leqslant \alpha$ and $y_2(u, t)$: $\alpha \leqslant u \leqslant 1$ where $u = x/\ell$ and $\alpha = a/\ell$. Both y_1 and y_2 have to obey the partial differential equation

$$\frac{\mathrm{EI}}{\ell^4}\frac{\partial^4 y_i(u, t)}{\partial u^4} + \mu\frac{\partial^2 y_i(u, t)}{\partial t^2} = 0; \quad i = 1, 2. \tag{2}$$

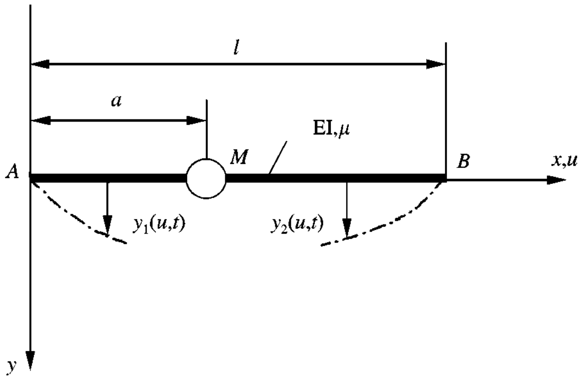

Figure 1. Beam–point mass system.

Assuming harmonic motion of the form

$$y_i(u, t) = Y_i(u)\cos\omega t, \quad i = 1, 2, \tag{3}$$

one obtains the general solution

$$Y_i(u) = C_{i1}\sin\lambda u + C_{i2}\cos\lambda u + C_{i3}\sinh\lambda u + C_{i4}\cosh\lambda u, \quad i = 1, 2 \tag{4}$$

for the space dependence of the motion, where λ is defined by

$$\omega = \sqrt{\frac{\mathrm{EI}}{\mu\ell^4}}\lambda^2. \tag{5}$$

This solution must satisfy two boundary conditions (which may easily be written once the end conditions be specified) for each end, and the four following matching conditions:

$$y_1(\alpha, t) = y_2(\alpha, t), \qquad y_1'(\alpha, t) = y_2'(\alpha, t) \qquad y_1''(\alpha, t) = y_2''(\alpha, t),$$

$$\frac{\mathrm{EI}}{\ell^3}[y_1'''(\alpha, t) - y_2'''(\alpha, t)] - M\ddot{y}_{1,2}(\alpha, t) = 0, \tag{6}$$

where primes denote differentiation with respect to u and overdots denote differentiation with respect to t. These conditions constitute, for the eight constants C_{ij} of equation (4), a set of eight homogeneous equations, the solvability condition of which yields the exact frequency equation of the system in question.

On the other hand, assuming again harmonic motion of form (3) and forming the ratio of the maximal potential energy to the so-called reference kinetic energy, the Rayleigh quotient of the system may be written

$$R = \frac{\mathrm{EI}}{\mu\ell^4}\lambda_R^4 \tag{7}$$

with

$$\lambda_R^4 = \frac{\int_0^\alpha Y_1''^2(\xi)\,\mathrm{d}\xi + \int_\alpha^1 Y_2''^2(\xi)\,\mathrm{d}\xi}{\int_0^\alpha Y_1^2(\xi)\,\mathrm{d}\xi + \int_\alpha^1 Y_2^2(\xi)\,\mathrm{d}\xi + \gamma Y_{1,2}^2(\alpha)}, \tag{8}$$

where $\gamma = M/\mu\ell$ (ratio of the point mass to the beam mass) and ξ is a dummy variable. As is well known, if good estimates corresponding to the fundamental mode shape are substituted for $Y_1(u)$ and $Y_2(u)$ into equation (8), the Rayleigh quotient of equation (7) approaches (from above) the square of the fundamental frequency, the closeness being warranted by the Rayleigh Principle [1, v.1. p 109] which states in today's terminology, that the Rayleigh quotient (viewed as a function of the shape function constraining the motion) is stationary in value when the shape function equals that of a natural mode. Thus, one has

$$\omega \cong \sqrt{\frac{\mathrm{EI}}{\mu\ell^4}}\lambda_R^2. \tag{9}$$

3. CASE STUDIES

In this section, exact frequency equations and Rayleigh approximations will be derived for various beam–point mass systems and their results be compared.

3.1. SIMPLY SUPPORTED BEAM

For a simply supported beam the boundary conditions are $y_1(0, t) = y_1''(0, t) = y_2(1, t) = y_2''(1, t) = 0$, which, together with equation (6) yield the frequency equation

$$2s\,\lambda\,\mathrm{sh}\,\lambda + \gamma\lambda(\mathrm{s}\,\lambda\,\mathrm{sh}\,\alpha\lambda\,\mathrm{sh}\,\beta\lambda - \mathrm{sh}\,\lambda\,\mathrm{s}\,\alpha\lambda\,\mathrm{s}\,\beta\lambda) = 0 \tag{10}$$

as also given by Lau [5] and Özkaya *et al.* [6]. Here $\beta = 1 - \alpha$ and the abbreviations s, c, sh, ch are used for sin, cos, sinh, cosh respectively. As can easily be verified, this equation reduces, at it should do, to the well-known frequency equation of a bare simply supported beam when either $\gamma = 0$, $\alpha = 0$ or $\alpha = 1$. On the other hand, it can easily be shown that the statically deflected shape of the same beam acted upon by its own weight and that of the point mass (see for example reference [7]), is given by

$$\bar{Y}_1(u) = u - 2u^3 + u^4 + 4\gamma\beta\lfloor(1 + \beta)\alpha u - u^3\rfloor$$

$$\bar{Y}_2(u) = u - 2u^3 + u^4 + 4\gamma\alpha\lfloor(1 + \alpha)\beta(1 - u) - (1 - u)^3\rfloor \tag{11}$$

to within a constant multiplier. Substituting equation (11) into equation (8) and performing the necessary calculations, one obtains

$$\lambda_R^4 = 3024\frac{1 + 10\gamma\delta[1+(1+4\gamma)\delta]}{31+6\gamma\delta\{3[17+4\delta(13+19\delta+9\delta^2)] + 16\gamma\delta[8 + 121\delta + 3(39 + 140\gamma)\delta^2]\}}, \tag{12}$$

where $\delta = \alpha\beta$. Substitution of this value of λ_R into equation (9) gives the approximate Rayleigh formula sought for the fundamental frequency. We note that equation (12) is equivalent to its counterpart in reference [4]. But a much simpler form is achieved here by introducing the variable δ to replace α, this being suggested by the very nature of the problem whose symmetry requires interchangeability of α and $\beta = 1 - \alpha$. Notice that the frequency equation (10) also reflects this feature.

In order to check the accuracy of this formula, the λ_R^2 values calculated from equation (12) are compared in Table 1 with the exact λ^2 values found by numerically solving equation (10), for different values of the parameters γ and α. The λ_{RT}^2 values corresponding to the well-known formula due to Timoshenko [2, p. 39] which, in the terminology of this note gives

$$\lambda_{RT}^4 = \frac{315}{3(1 + 35\gamma)\delta^2 + 2(2\delta + 1)} \tag{13}$$

TABLE 1

Simply supported beam

[Entries: (1) λ^2, (2) λ_R^2, (3) λ_{RT}^2, (4) % Error of λ_R^2, □ 0–0·1, ▒ 0·1–1, ■ > 1]

$\gamma \backslash \alpha$	0·001	0·01	0·1	0·2	0·3	0·4	0·5
0·001	9·869604 9·876659 12·537372 0·071	9·869595 9·876655 12·426544 0·072	9·868662 9·875743 11·492047 0·072	9·866196 9·873249 10·762464 0·071	9·86315 9·870181 10·286578 0·071	9·860689 9·867724 10·017749 0·071	9·859749 9·866792 9·930785 0·071
0·01	9·869603 9·876664 12·537369 0·072	9·869507 9·876623 12·426262 0·072	9·860186 9·867515 11·473651 0·074	9·835632 9·842673 10·714911 0·072	9·805553 9·812381 10·215324 0·07	9·781471 9·788341 9·932001 0·07	9·772337 9·77927 9·840222 0·071
0·1	9·869595 9·876713 12·53734 0·072	9·868631 9·87635 12·423442 0·078	9·776007 9·786564 11·29442 0·108	9·54102 9·54796 10·271632 0·073	9·274631 9·27984 9·575683 0·056	9·078467 9·084021 9·180342 0·061	9·007819 9·013681 9·052691 0·065
0·25	9·86958 9·876797 13·53729 0·073	9·86717 9·8761 12·418747 0·09	9·638107 9·655786 11·013506 0·183	9·092259 9·098829 9·641227 0·072	8·540967 8·54455 8·733417 0·042	8·175939 8·180058 8·235155 0·05	8·051867 8·056228 8·077296 0·054
0·5	9·869556 9·876944 12·537209 0·075	9·864736 9·876223 12·410933 0·116	9·415187 9·445297 10·58856 0·32	8·447616 8·453087 8·807436 0·065	7·613939 7·616165 7·718413 0·029	7·123625 7·126302 7·152828 0·038	6·96598 6·968717 6·978227 0·039
1	9·869507 9·877263 12·537045 0·079	9·859869 9·878305 12·395349 0·187	8·996193 9·043229 9·868055 0·532	7·454133 7·457472 7·634234 0·045	6·394693 6·39582 6·436777 0·018	5·846841 5·84816 5·857537 0·023	5·679598 5·680866 5·683986 0·022
2	9·86941 9·877998 12·536717 0·087	9·850138 9·888507 12·364355 0·39	8·264076 8·31604 8·780986 0·629	6·172948 6·174268 6·238869 0·021	5·061681 5·062124 5·074386 0·009	4·545023 4·545495 4·548021 0·01	4·393144 4·393571 4·394354 0·01
4	9·869215 9·879849 12·538061 0·108	9·830692 9·925039 12·303061 0·96	7·153189 7·187029 7·378274 0·473	4·817799 4·818156 4·83605 0·007	3·825118 3·82525 3·828191 0·003	3·394598 3·394726 3·395289 0·004	3·270907 3·271018 3·271183 0·003
10	9·86863 9·888165 12·534096 0·198	9·772477 10·07354 12·124504 3·081	5·332202 5·340998 5·378427 0·165	3·259863 3·259909 3·262368 0·001	2·527936 2·527956 2·528319 0	2·225188 2·225206 2·225272 0	2·139513 2·139527 2·139546 0
100	9·859865 10·218392 12·504721 3·636	8·936876 9·749675 10·131107 9·095	1·897839 1·897889 1·898079 0·003	1·077093 1·077093 1·077102 0	0·82215 0·82215 0·822152 0	0·719826 0·719826 0·719826 0	0·691144 0·691144 0·691144 0
1000	9·772391 11·542111 12·221899 18·109	5·012376 5·048546 5·054255 0·722	0·607729 0·607729 0·607729 0	0·342154 0·342154 0·342154 0	0·260737 0·260737 0·260737 0	0·228159 0·228159 0·228159 0	0·219036 0·219036 0·219036 0

are also shown in the table. The four entries of the table are, respectively, the λ^2, λ_R^2, λ_{RT}^2 values and the percentage error of a frequency calculation based on λ_R. To facilitate interpreting the validity limits of formula (12), different shading are applied to the regions corresponding to different percentage error intervals. Upon inspection of this table one concludes that: (1) Formula (13) can be reliably used in but a limited parameter region centred about the bottom right corner of the table. (2) The performance of formula (12) is satisfactory throughout the parameter space (except a limited region centred about the bottom left corner of the table) and is always superior, as also noted in reference [4], to that of formula (13). (3) Formula (12) (whence the used mode shape estimate) falls in qualitative disaccord with the truth (λ^2 should monotonically increase when moving upward and leftward on the table) in the region where $\alpha < 0{\cdot}1$ (and $\alpha > 0{\cdot}9$). (4) Formula (12) can safely be used over the range $0{\cdot}1 \leqslant \alpha \leqslant 0{\cdot}9$ with qualitative accuracy and a quantitative error of generally less than 0·1% and never higher than 1%.

3.2. FIXED–FIXED BEAM

Now, the boundary conditions are $y_1(0, t) = y_1'(0, t) = y_2(1, t) = y_2'(1, t) = 0$ and the corresponding frequency equation is

$$2(1 - \mathrm{c}\,\lambda\,\mathrm{ch}\,\lambda) + \gamma\lambda(\mathrm{s}\lambda\,\mathrm{ch}\,\alpha\lambda\,\mathrm{ch}\,\beta\lambda - \mathrm{sh}\,\lambda\,\mathrm{c}\,\alpha\lambda\,\mathrm{c}\,\beta\,\lambda + \mathrm{c}\,\alpha\lambda\,\mathrm{sh}\,\alpha\lambda + \mathrm{c}\,\beta\lambda\,\mathrm{sh}\,\beta\lambda$$
$$- \mathrm{s}\,\alpha\lambda\,\mathrm{ch}\,\alpha\lambda - \mathrm{s}\,\beta\lambda\,\mathrm{ch}\,\beta\lambda) = 0. \qquad (14)$$

One may easily verify that this equation reduces to that of a bare fixed–fixed beam when $\gamma = 0$, $\alpha = 0$ or $\alpha = 1$, and that it is invariant under permutation of α and β. The equation of the static deflection curve may be shown to be

$$\bar{Y}_1(u) = u^2 - 2u^3 + u^4 + 4\gamma\beta^2 u^2[3\alpha - (1 + 2\alpha)u],$$

$$\bar{Y}_2(u) = u^2 - 2u^3 + u^4 + 4\gamma\alpha^2(1 - u)^2[3\beta - (1 + 2\beta)(1 - u)] \qquad (15)$$

to within a scaling factor. Substituting equations (15) into equation (8), one obtains

$$\lambda_R^4 = 504\frac{1 + 60\gamma\delta^2(1 + 4\gamma\delta)}{1 + 6\gamma\delta^2\{9 + 4\delta + 108\delta^2 + 48\gamma\delta^2[3(1 + 12\delta) + 140\gamma\delta^2]\}}, \qquad (16)$$

which is again equivalent to, but much simpler than its counterpart in reference [4]. The results of the equations (14) and (16) are compared in Table 2 where the percentage error of the latter is also given. An inspection of this table shows that although equation (16) loses its qualitative accuracy in the region where $\alpha < 0{\cdot}2$ (and $\alpha > 0{\cdot}8$), it can be reliably used in the range $0{\cdot}2 \leqslant \alpha \leqslant 0{\cdot}8$ with always less than 1% of error.

TABLE 2

Fixed–fixed beam

[Entries: (1) λ^2, (2) λ_R^2, (3) % Error of λ_R^2, □ 0–0·1, □ 0·1–1, ■ > 1]

$\gamma\backslash\alpha$	0·001	0·01	0·1	0·2	0·3	0·4	0·5
0·001	22·373285 22·449944 0·343	22·373285 22·449951 0·343	22·372885 22·449817 0·344	22·368994 22·445798 0·343	22·359855 22·436281 0·342	22·349621 22·426 0·342	22·345121 22·421582 0·342
0·01	22·373285 22·449945 0·343	22·373285 22·450007 0·343	22·369283 22·448733 0·355	22·33035 22·40848 0·35	22·239598 22·313959 0·334	22·139542 22·213494 0·334	22·096131 22·170823 0·338
0·1	22·373285 22·449951 0·343	22·37328 22·450596 0·346	22·333025 22·44362 0·495	21·942775 22·0342 0·417	21·099603 21·1564 0·269	20·293398 20·349271 0·275	19·979493 20·038211 0·294
0·25	22·373285 22·449961 0·343	22·373272 22·451666 0·35	22·271668 22·453693 0·817	21·297051 21·403384 0·499	19·448569 19·486415 0·195	17·985526 18·023964 0·214	17·476082 17·515724 0·227
0·5	22·373285 22·449978 0·343	22·373258 22·453696 0·36	22·16682 22·501403 1·509	20·244974 20·35089 0·523	17·286163 17·30845 0·149	15·397744 15·420754 0·15	14·799958 14·822216 0·15
1	22·373285 22·450014 0·343	22·373231 22·458675 0·382	21·947429 22·615383 3·04	18·335998 18·406004 0·382	14·40298 14·413827 0·075	12·404729 12·415005 0·083	11·818212 11·827329 0·077
2	22·373285 22·450089 0·343	22·373177 22·47227 0·443	21·470762 22·593991 5·231	15·454863 15·479388 0·159	11·284045 11·288242 0·037	9·495895 8·499249 0·035	8·99446 8·997223 0·031
4	22·373285 22·450255 0·344	22·373068 22·513693 0·629	20·399196 21·524991 5·571	12·12309 12·12833 0·043	8·453546 8·454785 0·015	7·019395 7·020247 0·012	6·626572 6·62724 0·01
10	22·373285 22·450881 0·347	22·372742 22·745148 1·665	16·997361 17·330529 1·96	8·188118 8·18866 0·007	5·54769 5·547869 0·003	4·567376 4·567486 0·002	4·302516 4·302599 0·002
100	22·373285 22·483195 0·491	22·367773 34·453302 54·031	6·325314 6·326284 0·015	2·694419 2·694421 0	1·795165 1·795166 0	1·470188 1·470188 0	1·383074 1·383074 0
1000	22·37328 24·914718 11·359	22·309169 47·494544 112·892	2·025831 2·025831 0	0·855439 0·855439 0	0·569009 0·569009 0	0·465754 0·465754 0	0·438097 0·438097 0

3.3. FIXED–HINGED BEAM

In the case where the beam of Figure 1 is fixed at A and hinged at B, so that the boundary conditions are $y_1(0,t) = y_1'(0,t) = y_2(1,t) = y_2''(1,t) = 0$, the frequency equation becomes

$$2(\mathrm{s}\,\lambda\,\mathrm{ch}\,\lambda - \mathrm{c}\,\lambda\,\mathrm{sh}\,\lambda) + \gamma\lambda[\mathrm{c}\,\alpha\lambda(\mathrm{sh}\,\lambda\,\mathrm{s}\,\beta\lambda - \mathrm{ch}\,\lambda\,\mathrm{c}\,\beta\lambda) + \mathrm{ch}\,\alpha\lambda(\mathrm{s}\,\lambda\,\mathrm{sh}\,\beta\lambda + \mathrm{c}\,\lambda\,\mathrm{ch}\,\beta\lambda) - 2\mathrm{s}\,\beta\lambda\,\mathrm{sh}\,\beta\lambda] = 0, \quad (17)$$

Table 3

Fixed–hinged beam (*Entries as in Table* 2)

$\gamma\backslash\alpha$	0·001	0·01	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	0·99	0·999
0·001	15·418206 15·451111 0·213	15·418206 15·451114 0·213	15·418067 15·451129 0·214	15·416605 15·4497 0·215	15·412657 15·445562 0·213	15·406989 15·439753 0·213	15·402136 15·434918 0·213	15·400761 15·433595 0·213	15·40386 15·436684 0·213	15·409959 15·442782 0·213	15·415812 15·448704 0·213	15·418181 15·451091 0·213	15·418205 15·451112 0·213
0·01	15·418206 15·451112 0·213	15·418206 15·451141 0·214	15·416821 15·451316 0·224	15·402189 15·437011 0·226	15·36285 15·395738 0·214	15·306933 15·338457 0·206	15·259619 15·291321 0·208	15·246286 15·278478 0·211	15·276317 15·308427 0·21	15·33615 15·368243 0·209	15·394288 15·427085 0·213	15·417954 15·450911 0·214	15·418203 15·451114 0·213
0·1	15·418206 15·451114 0·213	15·418204 15·451411 0·215	15·404319 15·455518 0·332	15·257596 15·310853 0·349	14·87851 14·910252 0·213	14·388621 14·410653 0·153	14·01543 14·038871 0·167	13·914423 13·940621 0·188	14·139973 14·166268 0·186	14·637876 14·663741 0·177	15·180974 15·214819 0·223	15·415693 15·449303 0·218	15·418181 15·451142 0·214
0·25	15·418206 15·451119 0·213	15·418201 15·451889 0·218	15·383316 15·470861 0·569	15·015409 15·099235 0·558	14·129332 14·156677 0·194	13·135711 13·149155 0·102	12·46907 12·484526 0·124	12·295879 12·314404 0·151	12·67753 12·69717 0·155	13·622962 13·641628 0·137	14·833853 14·873334 0·266	15·411925 15·447367 0·23	15·418143 15·451199 0·214
0·5	15·418206 15·451127 0·214	15·418197 15·452763 0·224	15·347854 15·514144 1·038	14·611404 14·7329 0·832	13·040975 13·059938 0·145	11·586214 11·593494 0·063	10·730146 10·738971 0·082	10·515269 10·526225 0·104	10·980916 10·993634 0·116	12·269039 12·280806 0·096	14·281269 14·329811 0·34	15·405643 15·446066 0·262	15·41808 15·451318 0·216
1	15·418206 15·451143 0·214	15·418188 15·454795 0·237	15·275205 15·63739 2·371	13·820332 13·961795 1·024	11·368269 11·376869 0·076	9·609282 9·612476 0·033	8·69768 8·70147 0·044	8·47491 8·479656 0·056	8·948171 8·954322 0·069	10·396389 10·40222 0·056	13·281521 13·333485 0·391	15·393078 15·449784 0·368	15·417954 15·451649 0·219
2	15·418206 15·451176 0·214	15·41817 15·459991 0·271	15·122982 15·887264 5·054	12·397541 12·49575 0·792	9·265502 9·267935 0·026	7·520421 7·521542 0·015	6·694206 6·695418 0·018	6·495413 6·49692 0·023	6·910308 6·912452 0·031	8·273318 8·275548 0·027	11·680893 11·716836 0·308	15·367941 15·476618 0·707	15·417703 15·452677 0·227
4	15·418206 15·451247 0·214	15·418134 15·474849 0·368	14·791343 18·017776 8·292	10·311649 10·347138 0·344	7·127253 7·12777 0·007	5·63744 5·637751 0·006	4·967031 4·967337 0·006	4·806918 4·807294 0·008	5·136906 5·137475 0·011	6·266963 6·267622 0·011	9·586033 9·60044 0·15	15·317648 15·574324 1·676	15·4172 15·456125 0·252
10	15·418206 15·451501 0·216	15·418027 15·553813 0·881	13·626813 14·562811 6·869	7·286102 7·290805 0·065	4·763366 4·763425 0·001	3·70346 3·703503 0·001	3·240812 3·240851 0·001	3·130749 3·130796 0·002	3·355488 3·355563 0·002	4·145886 4·145983 0·002	6·743329 6·745692 0·035	15·16669 15·905985 4·874	15·415692 15·476336 0·393
100	15·418206 15·462336 0·286	15·416408 20·367484 32·116	5·99995 6·005825 0·098	2·465188 2·465202 0·001	1·55956 1·55956 0	1·199545 1·199545 0	1·04512 1·04512 0	1·008456 1·008456 0	1·082847 1·082847 0	1·348811 1·348811 0	2·286112 2·286122 0	13·029013 13·955299 7·109	15·393062 16·347986 6·204
1000	16·418204 16·25069 5·399	15·39894 39·74672 158·113	1·944114 1·944132 0·001	0·784771 0·784771 0	0·494924 0·494924 0	0·380259 0·380259 0	0·331158 0·331158 0	0·319502 0·319502 0	0·343135 0·343135 0	0·42777 0·42777 0	0·728116 3·728116 0	6·092207 6·107266 0·247	15·16623 18·546974 22·291

which is equivalent to that given in reference [6] and which reduces to that of a bare fixed–hinged beam when $\gamma = 0$, $\alpha = 0$ or $\alpha = 1$. The statically deflected shape is now

$$\bar{Y}_1(u) = 3u^2 - 5u^3 + 2u^4 + 4\gamma u^2 \lfloor 3\alpha(\alpha^2 - 3\alpha + 2) - (\alpha^3 - 3\alpha^2 + 2)u \rfloor,$$

$$\bar{Y}_2(u) = 3u^2 - 5u^3 + 2u^4 + 4\gamma \alpha^2 \lfloor 3u(u^2 - 3u + 2) - (u^3 - 3u^2 + 2)\alpha \rfloor, \quad (18)$$

which, when substituted into equation (8) gives

$$\lambda_R^4 = 1512 \frac{3 + 40\gamma\alpha\delta[3 - 2\alpha + 2(4 - \alpha)\gamma\delta]}{\{19 - 18\gamma\alpha\delta\{144\alpha^5 - 576\alpha^4 + 747\alpha^3 - 303\alpha^2 + 12\alpha - 38 + 8\gamma\alpha\delta[3(24\alpha^4 - 157\alpha^3 + 271\alpha^2 - 124\alpha - 16) - 70\gamma\delta^2(4 - \alpha)^2]\}\}}. \quad (19)$$

The results of equations (17) and (19) are compared in Table 3. Upon inspection of this table one concludes that equation (19) can safely be used over the range $0{\cdot}2 \leqslant \alpha \leqslant 0{\cdot}9$ with qualitative accuracy and a quantitative error of at most 1%.

3.4. CANTILEVER BEAM

In this case where the beam of Figure 1 is fixed at A and free at B, the boundary conditions are $y_1(0, t) = y_1'(0, t) = y_2''(1, t) = y_2'''(1, t) = 0$ and the resulting frequency equation is

$$2(1 + \mathrm{c}\,\lambda\,\mathrm{ch}\,\lambda) + \gamma\lambda(\mathrm{sh}\,\lambda\,\mathrm{c}\,\alpha\lambda\,\mathrm{c}\,\beta\lambda - \mathrm{s}\,\lambda\,\mathrm{ch}\,\alpha\lambda\,\mathrm{ch}\,\beta\lambda + \mathrm{c}\,\alpha\lambda\,\mathrm{sh}\,\alpha\lambda - \mathrm{c}\,\beta\lambda\,\mathrm{sh}\,\beta\lambda$$

$$- \mathrm{s}\,\alpha\lambda\,\mathrm{ch}\,\alpha\lambda + \mathrm{s}\,\beta\lambda\,\mathrm{ch}\,\beta\lambda) = 0, \quad (20)$$

which reduces to that of a bare cantilever when $\gamma = 0$ or $\alpha = 0$ and to that of a cantilever carrying a tip mass [8] when $\alpha = 1$ ($\beta = 0$). The statically deflected shape is now

$$\bar{Y}_1(u) = 6u^2 - 4u^3 + u^4 + 4\gamma u^2(3\alpha - u),$$

$$\bar{Y}_2(u) = 6u^2 - 4u^3 + u^4 + 4\gamma\alpha^2(3u - \alpha) \quad (21)$$

so that equation (8) gives

$$\lambda_R^4 = 756 \frac{3 + 5\gamma\alpha^2 \lfloor 6 + 4(\gamma - 1)\alpha + \alpha^2 \rfloor}{\{182 + 9\gamma\alpha^2\{182 - 84\alpha + 315\alpha^2 - 420\alpha^3 + 252\alpha^4 - 72\alpha^5 + 9\alpha^6 + 4\gamma\alpha^2[3(35 + 35\alpha - 35\alpha^2 + 11\alpha^3) + 140\gamma\alpha^2]\}\}} \quad (22)$$

which is the same as its counterpart in reference [4]. The results of equations (20) and (22) are compared in Table 4 where the results of Lord Rayleigh's formula (1) are also shown (on the last column). It is interesting to note that the accuracy of

TABLE 4

Cantilever Beam (Entries as Table 2, last column Lord Rayleigh's formula)

$\gamma\backslash\alpha$	0·001	0·01	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1	
0·001	3·516015 3·53009 0·4	3·516015 3·530091 0·4	3·516013 3·530102 0·401	3·515987 3·530097 0·401	3·515884 3·530005 0·402	3·515644 3·529756 0·401	3·515205 3·529291 0·401	3·514521 3·528574 0·4	3·513563 3·527587 0·399	3·51232 3·526331 0·399	3·510797 3·524815 0·399	3·509003 3·52305 0·4	3·559987
0·01	3·516015 3·53009 0·4	3·516015 3·530092 0·4	3·515995 3·530202 0·404	3·515728 3·530155 0·41	3·514706 3·529239 0·413	3·512303 3·526744 0·411	3·507933 3·522114 0·404	3·501151 3·515006 0·396	3·491714 3·505296 0·389	3·479577 3·49303 0·387	3·46484 3·478355 0·39	3·447658 3·461437 0·4	3·494181
0·1	3·516015 3·530091 0·4	3·516015 3·530108 0·401	3·515817 3·531268 0·439	3·513145 3·530941 0·506	3·502941 3·521734 0·536	3·479242 3·496827 0·505	3·43725 3·452132 0·433	3·374899 3·386939 0·357	3·293246 3·303307 0·305	3·195576 3·20483 0·29	3·085916 3·095421 0·308	2·967838 2·978501 0·359	2·989343
0·25	3·516015 3·530091 0·4	3·516015 3·530134 0·402	3·51552 3·53328 0·505	3·508835 3·532966 0·688	3·483423 3·509624 0·752	3·425531 3·447628 0·645	3·3272 3·342443 0·458	3·190552 3·200149 0·301	3·025831 3·03236 0·216	2·845588 2·851079 0·193	2·660222 2·665885 0·213	2·476693 2·483286 0·266	2·485251
0·5	3·516015 3·530091 0·4	3·516015 3·530179 0·403	3·515025 3·537255 0·632	3·501639 3·537873 1·035	3·451156 3·489587 1·114	3·339803 3·367223 0·821	3·162826 3·177346 0·459	2·938862 2·945616 0·23	2·695245 2·69887 0·134	2·453412 2·456185 0·113	2·225379 2·228201 0·127	2·016299 2·01957 0·162	2·019324
1	3·516015 3·530092 0·4	3·516015 3·530273 0·405	3·514032 3·547328 0·947	3·487201 3·550992 1·829	3·38769 3·446831 1·746	3·181884 3·213426 0·991	2·891238 2·902794 0·4	2·571727 2·575423 0·144	2·26652 2·268029 0·067	1·99427 1·995312 0·052	1·758689 1·759722 0·059	1·557298 1·558465 0·075	1·558122
2	3·516015 3·530094 0·4	3·516015 3·530479 0·411	3·512041 3·574501 1·778	3·458162 3·578805 3·489	3·265523 3·346997 2·495	2·913768 2·9419 0·965	2·50111 2·507821 0·268	2·119508 2·120991 0·07	1·799392 1·799864 0·026	1·539222 1·539521 0·019	1·328932 1·329221 0·022	1·158197 1·158513 0·027	1·158384
4	3·516015 3·530098 0·401	3·516014 3·530957 0·425	3·508032 3·647554 3·977	3·399581 3·606303 6·081	3·042437 3·124235 2·689	2·518698 2·535366 0·662	2·033591 2·03627 0·132	1·652684 1·653128 0·027	1·364099 1·364216 0·009	1·14452 1·144591 0·006	0·974887 0·974953 0·007	0·841546 0·841617 0·008	0·841584
10	3·516015 3·530109 0·401	3·516013 3·532922 0·481	3·495797 3·904874 11·702	3·222962 3·48313 8·072	2·535209 2·574702 1·558	1·882124 1·886473 0·231	1·42784 1·428332 0·034	1·119633 1·119697 0·006	0·904123 0·904138 0·002	0·74798 0·747989 0·001	0·631176 0·631184 0·001	0·541375 0·541383 0·001	0·541379
100	3·516015 3·530384 0·409	3·515994 3·648201 3·76	3·275201 3·997535 22·055	1·786952 1·798066 0·622	1·022642 1·023047 0·04	0·674353 0·67438 0·004	0·485676 0·485678 0	0·37069 0·370691 0	0·294714 0·294714 0	0·241491 0·241491 0	0·202527 0·202527 0	0·173001 0·173001 0	0·173001
1000	3·516015 3·543435 0·78	3·515799 7·112216 102·293	1·666299 1·674598 0·498	0·60769 0·607734 0·007	0·332322 0·332323 0	0·216176 0·216176 0	0·154784 0·154784 0	0·117788 0·117788 0	0·093489 0·093489 0	0·076528 0·076528 0	0·064139 0·064139 0	0·054766 0·054766 0	0·054766

that simple formula is better than that of equation (22) when $\gamma \geqslant 0{\cdot}5$. Upon inspection of the table, one concludes that equation (22) shows ill qualitative behaviour in the region where $\alpha \leqslant 0{\cdot}2$ but that it can safely be used with at most 1% error in the range $0{\cdot}4 \leqslant \alpha \leqslant 1$.

4. CONCLUDING REMARKS

It is shown that the Rayleigh formulae (12), (16), (19) and (22) can reliably predict the fundamental frequencies of the related beam–point mass systems, provided that they are used in proper parameter ranges. Rather conservative bounds are proposed here for these ranges. But the practiser may alter them by consulting Tables 1–4.

It may be supposed that, out of these formulae, especially those of equations (12) and (16) are simple enough to promote their use.

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